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NOTE ON A FORMULA OF SYMMETRIC FUNCTIONS.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

On page 8 of the German translation of Faa di Bruno's Formes Binaires, one reads : "The function $\phi_k[=\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_k{}^{p_k}]$ can be written symbollically in the form of a determinant as follows,

$$\phi_k=\begin{vmatrix} s_{p_1} & s_{(p_2)} & s_{(p_3)} & \cdots\cdots \\ s_{(p_2)} & s_{p_2} & s_{(p_2)} & \cdots\cdots \\ s_{(p_3)} & s_{(p_3)} & s_{p_3} & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix}$$

if after the development of the determinant, the symbolic products $s_{(p_1)}\,s_{(p_2)}\ldots\ldots$ are converted into $s_{p_1+p_2+\ldots\ldots}$."

If one endeavors to carry out these instructions, he finds that the statement holds for $k=2$, and $k=3$, but not farther. In the Annali di Tortolini (t. 5, 1854, p. 427-8), a clearer statement of this formula is given by Brioschi, but without proof. Brioschi's statement, slightly changing the notation, is as follows :

If $u_{rr}=s_{p_r}$, $u_{rs}u_{sr}=s_{p_r+p_s}$, $u_{rs}u_{st}u_{tr}=s_{p_r+p_s+p_t}$, and in general if every cycle of a substitution corresponds to a power sum whose index is the sum of those p's whose subscripts enter into the cycle, then the n ! products $u_{1r_1}u_{2r_2}u_{3r_3}\ldots\ldots u_{nr_n}$, corresponding to the $\begin{pmatrix}1 & 2 & 3\ldots\ldots n \\ r_1 & r_2 & r_3\ldots\ldots r_n\end{pmatrix}$, where the r's form a permutation of the numbers 1 2 3....n, give the terms of $\phi_n=\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}$, which we may write in the form of a determinant as follows :

$$\phi_n=\begin{vmatrix} u_{11} & u_{12}\cdots\cdots & u_{1n} \\ u_{21} & u_{22}\cdots\cdots & u_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u_{n1} & u_{n2}\cdots\cdots & u_{nn} \end{vmatrix}$$

It seems desirable to give here a proof of this theorem. One may be given in this manner: We form,

$$s_{p_n}\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_{n-1}^{p_{n-1}}=\sum_{r=1}^{r=n}\alpha_r{}^{p_n}(\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_{n-1}^{p_{n-1}}).$$

This gives $\phi_n=-\sum_{r=1}^{r=n-1}\phi_{n-1}^{(r)}+s_{p_n}\phi_{n-1}$, where $\phi_{n-1}^{(r)}$ is an abbreviation for $\Sigma\alpha_r^{p_n}(\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_{n-1}^{p_{n-1}})$. We assume that the determinant formula is true when $n-1$ roots are involved. Then the preceding equation takes the form

$$\phi_n=-\Sigma\begin{vmatrix} v_{11} & v_{12}\cdots\cdots & v_{1n-1} \\ v_{21} & v_{22}\cdots\cdots & v_{2n-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ v_{n-11} & v_{n-12}\cdots & v_{n-1n-1} \end{vmatrix}+u_{nn}\begin{vmatrix} u_{11} & u_{12}\cdots\cdots & u_{1n-1} \\ u_{21} & u_{22}\cdots\cdots & u_{2n-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u_{n-11} & u_{n-12}\cdots & u_{n-1n-1} \end{vmatrix}$$

The sum Σ contains $(n-1)$ determinants. Of these the r_{11}th one, which expresses $\phi_{n-1}^{(r)}$, must be similar to ϕ_{n-1}, but with this difference, that everywhere instead of p_r, p_r+p_n must occur. If we denote p_r+p_n by $p_r{}^1$, we may say, the symbol connected with $s_{p_r{}^1}$ must be so constituted as to always express $s_{p_r}+p_n$. Any product which corresponds to a cycle which begins and ends with r, that is, any product commencing with an element of the rth line, and ending with an element of the rth column, like

$$v_{rs}v_{st}\ldots\ldots v_{xy}v_{yr}=s_{p_r{}^1}+p_s+\ldots\ldots p_x+p_y$$

can be written in the u notation as

$$u_{nr}u_{rs}u_{st}\ldots\ldots u_{xy}u_{yn}=s_{p_r}+p_n+p_s+\ldots\ldots p_x+p_y,$$

since the right members of these two expressions are equal. As far as the products which correspond to the remaining cycles (which cannot involve r and n) are concerned, they may always be written simply with u's instead of v's. It follows: We may write the determinant for $\phi_{n-1}^{(r)}$ with u's instead of v's, provided that only v_{rs} goes over into $u_{nr}u_{rs}$, and v_{yr} into u_{yn}, and obtain exactly the same expansion as before. Writing $u_{nr}u_{r1}$, $u_{nr}n_{r2}\ldots\ldots u_{nr}u_{rn-1}$ for the rth line instead of v_{r1}, $v_{r2}\ldots\ldots v_{rn-1}$, and u_{1n}, $u_{2n}\ldots\ldots u_{n-1n}$ for the rth column instead of v_{1r}, $v_{2r}\ldots\ldots v_{n-1r}$, we have for $-\phi_{n-1}^{(r)}$, after taking out the factor u_{nr},

$$-\phi_{n-1}^{(r)} = -u_{nr} \begin{vmatrix} u_{11} & u_{12} \cdots\cdots u_{1r-1} & u_{1n} & u_{1r+1} \cdots\cdots u_{1n-1} \\ u_{21} & u_{22} \cdots\cdots u_{2r-1} & u_{2n} & u_{2r+1} \cdots\cdots u_{2n-1} \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ u_{r1} & u_{r2} \cdots\cdots u_{rr-1} & u_{rn} & u_{rr+1} \cdots\cdots u_{rn-1} \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ u_{n-11} & u_{n-12} \cdots\cdots\cdots\cdots & u_{n-1n} & \cdots\cdots\cdots u_{n-1n-1} \end{vmatrix}$$

In this determinant pass the rth column over the remaining $n-r-1$ columns. The entire sign factor becomes $(-1)^{n-r-1+1}=(-1)^{n-r-2}$. We can now combine our results conveniently as follows :

$$\phi_n = \sum_{r=1}^{r=n} (-1)^{n-r-2} u_{nr} \begin{vmatrix} u_{11} & u_{12} \cdots\cdots u_{1r-1} & u_{1r+1} \cdots\cdots u_{1n} \\ u_{21} & u_{22} \cdots\cdots u_{2r-1} & u_{2r+1} \cdots\cdots u_{2n} \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ u_{n-11} & u_{n-12} \cdots u_{n-1r-1} & u_{n-1r+1} \cdots u_{n-1n} \end{vmatrix}$$

if for $r=1$ the first column, and for $r=n$, the last column is suppressed. But the sum of the determinants on the right hand is none other than the expansion of

$$\begin{vmatrix} u_{11} & u_{12} \cdots\cdots u_{1n} \\ u_{21} & u_{22} \cdots\cdots u_{2n} \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ u_{n1} & u_{n2} \cdots\cdots u_{nn} \end{vmatrix}$$

in terms of the elements of the last line.

We have therefore proved that if ϕ_{n-1} can be expressed as a determinant of order $n-1$, ϕ_n can be similarly expressed as a determinant of order n. But the assumption is true when $n-1=1$, and therefore for all greater values.

Attention may be called to the fact that this determinant gives very beautifully the general term of ϕ_n in terms of the s's. Let σ be the number of cycles, involving $n_1, n_2, \ldots\ldots u_\sigma$ numbers respectively, where $n_2 + n_2 + \ldots\ldots n_\sigma = n$, into whose product any given substitution of the numbers 1, 2, 3 $\ldots\ldots n$ can be decomposed. The general term of ϕ_n will consist of the products which correspond to the σ cycles of the substitution. The modulus of the substitution will be the product of the moduli of the cycles, viz, $(-1)^{n_1-1+n_2-1+\ldots\ldots, n_\sigma-1}=(-1)^{n-\sigma}$. The coefficient of the term will be the number of ways in which the substitution can occur. The first cycle can occur in $(n_1-1)!$ ways, for for the first number the (n_1-1) remaining numbers may be substituted, for the second the (n_1-2) remaining, etc.; for the next to the last, the last remaining unused number, and for the last the first number. Similarly, a second cycle can occur in $(n_2-1)!$ ways, and the two together in $(n_1-1)!\,(n_2-1)!$ ways. We thus get for the general term of ϕ_n, if $\lambda_{n_1}, \lambda_{n_2}, \ldots\ldots \lambda_{n_\sigma}$ denote the sum of the $n_1, n_2, \ldots\ldots n_\sigma$

exponents respectively,

$$(-1)^{n-\sigma}(n_1-1)!\,(n_2-1)!\ldots\ldots(n\sigma-1)!\ {}^{s}\lambda_{n_1}\ {}^{s}\lambda_{n_2}\ \ldots\ldots{}^{s}\lambda_{n\sigma},$$

a result that agrees with that obtained in a somewhat different way on page 8 of the German translation of Faa di Bruno's Formes Binaires.

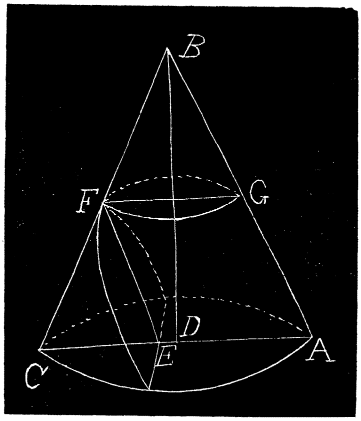
Erlangen, Bavaria, 4 May, 1898.

CONVEX SURFACE AND VOLUME OF CONICAL UNGULÆ.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let $BD=c$, $DE=a$, $\tan DBC=n$, $\cot FEC=m$. Also let $DH=h$, $DC=R$, $HF=r$, $EC=d$, then $c=\dfrac{Rh}{R-r}$, $n=\dfrac{R-r}{h}$, $a=R-d$, $m=\dfrac{r-R+d}{h}$.

Then $x^2+z^2=n^2(c-y)^2$, is the equation of the cone, and $x=my+a$, is the equation of the plane.



$$\left(\frac{dz}{dx}\right)^2=\frac{x^2}{z^2},\quad \left(\frac{dz}{dy}\right)^2=\frac{n^4(c-y)^2}{z^2};$$

$$\therefore \sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}$$

$$=\frac{n(c-y)\sqrt{(1+n^2)}}{\sqrt{[n^2(c-y)^2-x^2]}}.$$

The limits of x are $my+a=x_2$ and $n(c-y)=x_1$; of y, 0 and $\dfrac{nc-a}{m+n}=y'$.

$$\therefore\ S=2\int_0^{y'}\int_{x_2}^{x_1}\frac{n(c-y)\sqrt{(1+n^2)}}{\sqrt{[n^2(c-y)^2-x^2]}}.$$

$$=n\sqrt{(1+n^2)}\int_0^{y'}\left[\pi-2\sin^{-1}\left(\frac{a+my}{n(c-y)}\right)\right](c-y)dy$$

$$=nc^2\sqrt{(1-n^2)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{a}{nc}\right)\right]$$

$$-n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}.$$